Downside Performance Measures and the Sharpe Ratio

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Problem and Motivation

- Lower partial moments measure downside risk and inspired scholars to create different performance measures, which allow a large variation of their parameters, namely order *n* and target τ.
- Since mean-downside risk portfolio lines are non-linear in general, slope-related (target-dependent) downside performance measures may not be applicable unless the target is equal to the risk-free rate.
- This hypothesis is strengthened by several empirical analyses in which the rank order of mutual fund performance depends on the choice of the downside performance measure.
- Our understanding of performance of mutual funds relates to the portfolio line, which combines the positions of the mutual fund and the risk-free asset, since it is not the mutual fund itself, but rather its combination with the risk-free asset, that yields an optimal portfolio when maximizing expected utility.

Contribution to the Literature

We contribute to the literature on performance measurement as we provide a way to measure downside performance based on passive strategies, i.e., portfolio lines. Thereby, we:

- analyze the shape of mean-downside risk portfolio lines with arbitrary target above the risk-free rate in detail;
- show that the shape of these portfolio lines can be characterized by the Kappa ratio with a target equal to the risk-free rate – **asymptotic Kappa ratio** – and, surprisingly, the **Sharpe ratio** (Figures 1 and 2);
- show that minimum downside risk portfolios with varying target can be positioned on a straight line, so that its slope can serve as a target-independent performance measure – **minimum downside risk Kappa ratio** (Figure 3);
- conduct an empirical analysis of mutual funds performance to corroborate our analytical results.

Analytical Approach

Notation

Lower partial moment $\mathrm{LPM}_{n,\tau}(R) \equiv \int_{-\infty}^{\tau} (\tau - R)^n \, dF(R) = \mathrm{E}\left(\max\{\tau - R; 0\}^n\right)$

Portfolio rate of return $R_P = x \cdot R_M + (1-x) \cdot r$

Transformation of target return $\tau = x \cdot \varphi + (1-x) \cdot r \quad \Leftrightarrow \quad \varphi = r + \frac{\tau - r}{x}$

1. A reformulation of lower partial moment (I) and portfolio return (II) yields a functional form of the portfolio line (III) in $\left(\mathrm{E}(R), \sqrt[n]{\mathrm{LPM}_{n,\tau}(R)}\right)$-space:

(I) $\mathrm{LPM}_{n,\tau}(R_P) \equiv \int_{-\infty}^{\varphi(x)} \left(\tau - x \cdot R_M - (1-x) \cdot r\right)^n dF(R_M) =$

$x^n \cdot \int_{-\infty}^{\varphi(x)} \left(\frac{\tau - x \cdot R_M - (1-x) \cdot r}{x}\right)^n dF(R_M) = x^n \int_{-\infty}^{\varphi(x)} \left(\varphi(x) - R_M\right)^n dF(R_M)$

$\Rightarrow x = \frac{\sqrt[n]{\mathrm{LPM}_{n,\tau}(R_P)}}{\sqrt[n]{\mathrm{LPM}_{n,\varphi(x)}(R_M)}}$

(II) $R_P = x \cdot R_M + (1-x) \cdot r$

$\Rightarrow x = \frac{\mathrm{E}(R_P) - r}{\mathrm{E}(R_M) - r}$

(III) As $\varphi(x) \xrightarrow{x \to +\infty} r$, this slope represents the **asymptotic Kappa ratio**:

$$\mathrm{E}(R_P) = r + \frac{\mathrm{E}(R_M) - r}{\sqrt[n]{\mathrm{LPM}_{n,\varphi(x)}(R_M)}} \cdot \sqrt[n]{\mathrm{LPM}_{n,\tau}(R_P)} \to \begin{cases} \left.\frac{d\mathrm{E}(R_P)}{d\sqrt[n]{\mathrm{LPM}_{n,\tau}(R_P)}}\right|_{x \to +\infty} = \frac{\mathrm{E}(R_M) - r}{\sqrt[n]{\mathrm{LPM}_{n,\tau}(R_M)}} = \kappa_{n,\tau}(R_M) \\ \left.\frac{d\mathrm{E}(R_P)}{d\sqrt[n]{\mathrm{LPM}_{n,\tau}(R_P)}}\right|_{x \to -\infty} = -\frac{\mathrm{E}(R_M) - r}{\sqrt[n]{\mathrm{UPM}_{n,\tau}(R_M)}} \end{cases}$$

2. While the slope at $x = 0$ approaches –1, the curvature at this point is indirectly proportional to the **Sharpe ratio**:

$$\text{(IV)} \left.\frac{\frac{d^2}{dx^2}\sqrt[n]{\mathrm{LPM}_{n,\tau}(R_P)}}{\frac{d\mathrm{E}(R_P)}{dx} \cdot \frac{d\mathrm{E}(R_P)}{dx}}\right|_{x \to 0} =$$

$$\frac{(\tau - r)^{1-n} \cdot (n-1) \cdot (\tau - r)^{n-2} \cdot \int_{-\infty}^{\infty} (r - R_M)^2 \, dF(R_M) - (n-1) \cdot (\tau - r)^{1-2n} \cdot (\tau - r)^{2n-2} \cdot \left(\mathrm{E}(R_M) - r\right)^2}{\left(\mathrm{E}(R_M) - r\right)^2}$$

$$= \frac{(n-1)}{(\tau - r)} \cdot \frac{\sigma_M^2}{\left(\mathrm{E}(R_M) - r\right)^2} = \frac{(n-1)}{(\tau - r)} \cdot \frac{1}{\mathrm{SR}_M^2} \text{ with } \mathrm{SR}_M = \frac{\mathrm{E}(R_M) - r}{\sigma}$$

3. Minimum lower partial moment (MLPM) portfolios can be positioned on a staight line, where the slope can be interpreted as **minimum downside risk Kappa ratio**:

(V) $\frac{d}{dx}\sqrt[n]{\mathrm{LPM}_{n,\tau}(R_P)} = \left(\mathrm{LPM}_{n,\tau}(R_P)\right)^{\frac{1}{n}-1} \cdot \int_{-\infty}^{\varphi(x)} \left(\tau - \left(x \cdot R_M + (1-x) \cdot r\right)\right)^{n-1} \cdot (r - R_M) \, dF(R_M) = 0$

$$\Rightarrow r = \int_{-\infty}^{\varphi(x)} R_M \cdot \frac{\left(\varphi(x) - R_M\right)^{n-1}}{\int_{-\infty}^{\varphi(x)} \left(\varphi(x) - R_M\right)^{n-1} dF(R_M)} dF(R_M)$$

The φ^* that solves (V) is unique, renamed to *c* and inserted into (III) to obtain (VI).

(VI) $\Rightarrow \mathrm{E}(R_{\mathrm{MLPM}}) = r + \frac{\mathrm{E}(R_M) - r}{\sqrt[n]{\mathrm{LPM}_{n,c}(R_M)}} \cdot \sqrt[n]{\mathrm{LPM}_{n,\tau}(R_{\mathrm{MLPM}})}.$

Graphical Illustration

Figure 1 *Portfolio line and "wedge" of asymptotes*

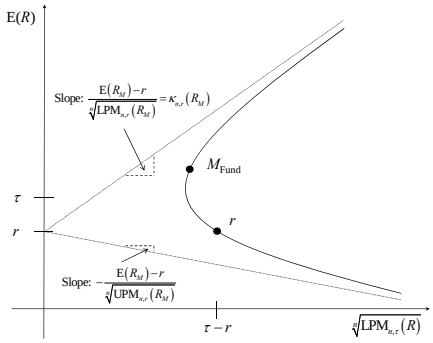


Figure 2 *Curvature at the risk-free asset and performance ranking*

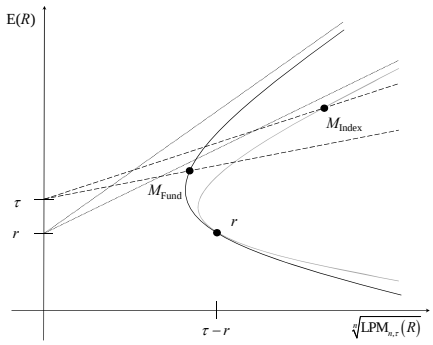
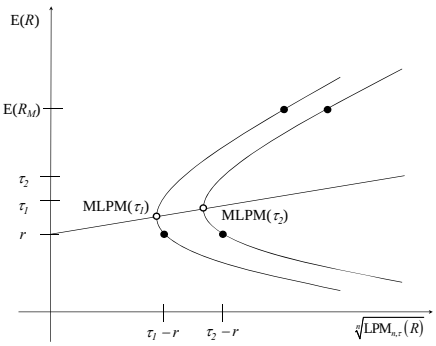


Figure 3 *Minimum downside risk portfolios*



Empirical Results

Descriptive Statistics	
Data source	Yahoo Finance
Number of mutual funds	73
Time period	01/2000 – 06/2019
Benchmark	S&P500
Risk-free rate (monthly)	0.08%
Target rate (monthly)	0.25% and 0.42%

Table 1 *Rank correlation of (downside) performance measures for U.S. mutual funds based on monthly returns (where τ = 0.25% per month)*

Rank correlation	SR	$\kappa_{2,\tau}$	$\kappa_{2,r}$	$\Omega(\tau)$	$\Omega_{2,r}$	$\frac{\mathrm{E}(R)-r}{\sqrt[2]{\mathrm{LPM}_{2,c}(R)}}$
SR		0.9680	0.9999	0.9660	0.9999	0.9946
$\kappa_{2,\tau}$			0.9690	0.9976	0.9690	0.9635
$\kappa_{2,r}$				0.9668	1.0000	0.9946
$\Omega(\tau)$					0.9690	0.9605
$\Omega_{2,r}$						0.9946

Table 2 *Rank correlation of (downside) performance measures for U.S. mutual funds based on monthly returns (where τ = 0.42% per month)*

Rank correlation	SR	$\kappa_{2,\tau}$	$\kappa_{2,r}$	$\Omega(\tau)$	$\Omega_{2,r}$	$\frac{\mathrm{E}(R)-r}{\sqrt[2]{\mathrm{LPM}_{2,c}(R)}}$
SR		0.9034	0.9999	0.9249	0.9999	0.9947
$\kappa_{2,\tau}$			0.9046	0.9901	0.9046	0.9005
$\kappa_{2,r}$				0.9258	1.0000	0.9946
$\Omega(\tau)$					0.9258	0.9192
$\Omega_{2,r}$						0.9946

Performance Measures	Description
SR	Sharpe ratio
$\kappa_{2,\tau}$	Kappa ratio of order 2 and target τ (Sortino ratio)
$\kappa_{2,r}$	Kappa ratio of order 2 and risk-free rate as target
$\Omega(\tau)$	Omega ratio of Shadwick and Keating (2002)
$\Omega_{2,r}$	Generalized Omega ratio of order 2 and risk-free rate as target
$\frac{\mathrm{E}(R)-r}{\sqrt[2]{\mathrm{LPM}_{2,c}(R)}}$	Slope of the MLPM portfolio line

Discussion / Implications

- Downside performance measures incorporate a more intuitive notion of risk. However, the classical Kappa ratio of a mutual fund can be levered and, thus, fails to meet the requirements for an admissible performance measure.
- Instead of the mutual fund itself, performance should be attributed to the portfolio line, i.e., the combinations of the mutual fund and the risk-free asset.
- **Asymptotic Kappa ratios** (with fixed target rate *r*) and the **Sharpe ratio** describe the shape of downside risk portfolio lines and, thus, are qualified to serve as (target-independent) downside performance measure.
- The analytical result of the MLPM portfolio line points to a direct (linear) connection between the investor's target rate and her optimal combination of mutual fund and risk-free asset.

Selected References

Farinelli, S., and L. Tibiletti (2008), Sharpe thinking in asset ranking with one-sided measures, *European Journal of Operational Research* 185, 1542–1547.

Fishburn, P.C. (1977), Mean-risk analysis with risk associated with below-target returns, *American Economic Review* 67(2), 116–126.

Harlow, W.V., and R.K.S. Rao (1989), Asset pricing in a general mean-lower partial moment framework: Theory and evidence, *Journal of Financial and Quantitative Analysis* 24, 285–311.

Hoechner, B., P. Reichling, and G. Schulze (2017), Pitfalls of downside performance measures with arbitrary targets, *International Review of Finance* 17, 597-610.

Kaplan, P.D., and J.A. Knowles (2004), Kappa: A generalized downside risk-adjusted performance measure, *Journal of Performance Measurement* 8(1), 42–54.

Pedersen, C.S., and S.E. Satchell (2002), On the foundation of performance measures under asymmetric returns, *Quantitative Finance*, 2, 217–223.

Shadwick, W.F., and C. Keating (2002), A universal performance measure, *Journal of Performance Measurement* 6(1), 59–84.