A Note on the Value Additivity of Certainty Equivalents

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Abstract

We show that, if value additivity is required, the use of certainty equivalents in company valuation implies both constant absolute and constant relative risk aversion. An investor only exhibits constant absolute and constant relative risk aversion at the same time when he is risk neutral. However, this in turn makes the use of certainty equivalents redundant.

Keywords

Company valuation, value additivity, certainty equivalent

JEL-Classification

G31, D81

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1. Preliminary Remarks

Company valuation from a financial point of view applies the net present value method under certainty to the valuation of risky cash flows. The main characteristic of the net present value under certainty reads as follows: The net present value is the monetary equivalent of the maximal possible additional consumption at the decision point in time $t = 0$ if the investment is realized. This characteristic is based upon the assumption of a perfect capital market and corresponds to the idea of duplication: The net present value can be realized additionally to the initial investment by a corresponding loan, in which the investment's cash flows ensure exactly its repayment. Therefore, the sum of discounted future cash flows indicates the maximum price, up to which the investment is profitable.

Analogously, the valuation of risky investments does not only request the determination of a fair value. In addition, the specification of a corresponding duplication strategy is desired. Ideally, the computed value of an investment should be achievable as net cash flow of the corresponding duplicating transactions. However, in general, this requires a complete capital market, on which duplicating portfolios can replicate all appearing cash flows. Then, the maximum price of the cash flow being valued corresponds to the price of the duplicating portfolio.

In particular for non-listed companies cash flows exist, which cannot be replicated on the capital market. In this case one assumes a model of alternative investments to determine the investors' required rate of return. For example, the capital asset pricing model (CAPM) provides the ex ante risk premiums claimed by equity investors. Here duplication only refers to the beta coefficient or the volatility of an efficient portfolio, which consists of the market portfolio and the risk-free investment according to this coefficient.

Since beta coefficients of non-listed companies cannot be estimated from a stock price time series, additionally or alternatively preference depending approaches are involved in valuation practice.[1] Instead of adding a risk premium to the risk-free interest rate, a deduction for risk is made here. Within this framework, the certainty-equivalent method proposes to compute the today's value V of a flow of risky, i.e. random, future cash flows $C_1, \ldots, C_T$ at points in time $t = 1, \ldots, T$ as follows:

[1] See IDW (2005)

$$(1)\quad V(C_1,\ldots,C_T)=\frac{\mathrm{CE}(C_1)}{1+r_1}+\cdots+\frac{\mathrm{CE}(C_T)}{(1+r_T)^T},$$

where the certainty equivalent $\mathrm{CE}(C)\equiv u^{-1}(\mathrm{E}(u(C)))$ is determined on the basis of the individual risk utility function u according to expected utility theory and r_1, …, r_T represent the spot rates. The procedure according to formula (1) requires extensive assumptions:

1) Firstly, it is necessary to determine simultaneously the certainty equivalent CE(C_1, …, C_T) of the entire risky cash flow. Formula (1) implies that the investor's risk utility function remains constant over time, which means it does not depend on the payment date.

2) Additionally, formula (1) assumes that the certainty equivalents of single payments can be transformed to the decision point in time by discounting and that these single values can be added up afterwards. This presumes certain properties of the risk utility function and the respective inverse function.[2]

3) For instance, one can determine the certainty equivalent of an exponential risk utility function, i.e. constant absolute risk aversion a, in case of normally distributed cash flows depending on the mean E and variance Var of the random cash flow in the following way: $\mathrm{CE}(C)=\mathrm{E}(C)-{}^{a}\!/_{2}\cdot\mathrm{Var}(C)$.[3] If cash flows are correlated, for the total variance Var(C) one has to consider the covariances of the single payments. Thus, formula (1) assumes uncorrelated cash flows.

4) By discounting with spot rates, formula (1) finally implies the existence of a perfect capital market for risk-free investments.

These assumptions are discussed controversially, especially in the German literature on company valuation.[4] We neither want to pick up this discussion in detail nor continue it. Instead, given the background of practical applications, we want to set up requirements for the valuation function given by the certainty equivalent. This is expressed formally by the properties of additivity and multiplicativity with respect to constants. From an economic point of view we

2 See Kürsten (2002)
3 See Bamberg/Spremann (1981)
4 See Diedrich (2003), Kruschwitz (2001), Kruschwitz/Löffler (2003), Kürsten (2002 and 2003), Schwetzler (2000, 2002a, and 2002b), and Wiese (2003)

request independence of the valuation from a risk-free payment as well as independence from the number of shares of a company being purchased.

To illustrate our argument we only discuss the valuation of a single risky cash flow at a certain date. Discounting the certainty equivalent of a future cash flow represents an additional issue, which will be discussed only marginally. We rather show that the valuation by means of certainty equivalents is problematic even without discounting if simple plausible requirements are made.

Our paper is organized as followings: Chapter 2 relates the mentioned requirements to the context of value additivity and contrasts them to the assumption of arbitrage-freeness. In chapter 3 we analyze whether our requirements to the valuation by certainty equivalents are fulfilled in case of constant absolute and constant relative risk aversion. The paper ends in chapter 4 with a short conclusion.

2. Value Additivity

Value additivity implies that the value of a sum of risky or risk-free cash flows is equal to the sum of their single values. Formally, the valuation function *V* exhibits the following property with regard to the cash flows *C* and *D*:[5]

(2) $V(C+D) = V(C) + V(D).$

For example, in case of a perfect capital market, the net present value formula under certainty follows the principle of value additivity. The CAPM equation also exhibits this property, since the beta coefficient of a portfolio equals the fraction-weighted sum of the beta coefficients of the single securities of which the portfolio consists. If we analyze additivity and multiplicativity with respect to constants *r* and *s* in the following, then due to this linearity we require a weak form of value additivity:

(3) $V(r + s \cdot C) = r + s \cdot V(C).$

5 See Schall (1972)

In neoclassical finance formula (3) represents a condition for arbitrage-freeness, since the latter assumes the existence of a positive linear valuation function according to the fundamental theorem of asset pricing.[6] Without this positivity only Jevon's law of one price is fulfilled. Hence, the required value additivity with respect to constants corresponds merely to this law. At the same time, the linearity of the valuation rule implies price taking behavior. In addition to the mathematical solution of equation (3),[7] our focus is to combine value additivity with respect to additive and multiplicative constants with certain classes of risk aversion.

3. Constant Absolute and Relative Risk Aversion

In order to identify the tight limits of the certainty equivalent method, it suffices to consider some special cases. We restrict ourselves to the class of risk utility functions with hyperbolic absolute risk aversion (HARA). For these utility functions it holds:

$$(4) \qquad -\frac{u''(C)}{u'(C)} = \frac{1}{\alpha + \beta \cdot C} \text{ where } C > -\frac{\alpha}{\beta}.$$

The solutions of differential equation (4) exhibit the form of an exponential, power or logarithmic function.[8] This includes the cases of constant absolute risk aversion (CARA) $a \equiv 1/\alpha > 0$ for $\beta = 0$ and constant relative risk aversion (CRRA) $\gamma \equiv 1/\beta \in (0,1)$ with $\beta \neq 1$ or $\gamma \equiv 1$ with $\beta = 1$ for $\alpha = 0$. We now examine the value, which investors with constant absolute or constant relative risk aversion would attribute to a risky future cash flow, and assume the following:

1) The company being valued pays only one positive risky cash flow C per share.

2) We consider three different types of investors who vary in terms of their risk aversion. Investor $I_{\exp}$ possesses constant absolute risk aversion a, i.e. an exponential risk utility function $u_{\exp}(C) = -e^{-a \cdot C}$. Investor I_{pow} exhibits constant relative risk aversion $\gamma \neq 1$. Therefore, he decides according to a power risk utility function $u_{\mathrm{pow}}(C) = 1/\gamma \cdot C^{1-\gamma}$. Fi-

6 See Ross (2004, p 1) and the references cited therein. This approach in company valuation is adopted by Wilhelm (2002) and Kruschwitz/Löffler (2005).
7 See Aczél (1987)
8 See Bamberg/Spremann (1981)

nally, investor $I_{\log}$ also shows constant relative risk aversion, but now amounting to one, which implies a logarithmic risk utility function $u_{\log}(C) = \ln C$. Table 1 provides an overview.

	Investor		
	$I_{\exp}$	I_{pow}	$I_{\log}$
Risk utility function	$-e^{-a \cdot C}$	$\frac{1}{\gamma} \cdot C^{1-\gamma}$	$\ln C$
Respective inverse function	$-\frac{1}{a} \cdot \ln(-C)$	$(\gamma \cdot C)^{1/(1-\gamma)}$	e^C

Tab. 1: Risk Utility Functions of the Investors

3.1 Valuation Independence from Additive Constants

An important practical requirement for the valuation calculus is that the value of a company must not turn out to be higher or lower because the investor receives or makes a risk-free payment beside the risky payment. For example, this is the case if the investor runs (partially) into debt to make the investment or if the investor holds a certain amount of cash beside the investment. In case of constant absolute risk aversion with respect to an additive constant *r* for the certainty equivalent we get:

$$\begin{aligned} \mathrm{CE}_{\exp}(C+r) &= -\frac{1}{a} \cdot \ln\left(-\mathrm{E}\left(-e^{-a \cdot (C+r)}\right)\right) \\ &= r - \frac{1}{a} \cdot \ln\left(-\mathrm{E}\left(-e^{-a \cdot C}\right)\right) \\ &= \mathrm{CE}_{\exp}(C) + r. \end{aligned} \tag{5}$$

Therefore, the value of a risky cash flow, which is computed using the certainty equivalent, does not change in case of constant absolute risk aversion if the investor receives or makes a risk-free payment from an additional transaction at the same time. For example, to finance the investment, the investor could take a credit with a repayment which is due at maturity and amounts to the certainty equivalent, so that in this case the value of a future risky cash flow can be determined by discounting the certainty equivalent with the risk-free spot rate with appropriate maturity.[9]

Rearranging equation (5) shows that in case of constant absolute risk aversion the risk premium RP, which equals the difference between expected value and certainty equivalent, does not depend on the initial wealth of the investor:

$$
\begin{aligned}
& \mathrm{CE}_{\exp}(C+r) = \mathrm{CE}_{\exp}(C) + r \\
(6) \qquad & \Leftrightarrow \mathrm{E}(C+r) - \mathrm{CE}_{\exp}(C+r) = \mathrm{E}(C+r) - \mathrm{CE}_{\exp}(C) - r \\
& \Leftrightarrow \mathrm{RP}_{\exp}(C+r) = \mathrm{RP}_{\exp}(C).
\end{aligned}
$$

On the other hand, this implies that in case of constant relative risk aversion the risk premium decreases with an increasing initial wealth. Hence, for $r > 0$ we get:[10]

$$
\begin{aligned}
& \mathrm{RP}_{\mathrm{pot}}(C+r) < \mathrm{RP}_{\mathrm{pot}}(C) \\
& \Leftrightarrow \mathrm{E}(C+r) - \mathrm{CE}_{\mathrm{pot}}(C+r) < \mathrm{E}(C+r) - \mathrm{CE}_{\mathrm{pot}}(C) - r \\
(7) \qquad & \Leftrightarrow \mathrm{CE}_{\mathrm{pot}}(C+r) > \mathrm{CE}_{\mathrm{pot}}(C) + r \\
& \text{or } \ \mathrm{CE}_{\log}(C+r) > \mathrm{CE}_{\log}(C) + r.
\end{aligned}
$$

In contrast to constant absolute risk aversion, in case of constant relative risk aversion the value of a risky cash flow, which is computed using the certainty equivalent, increases with the initial wealth of the investor. If we require that the company value must not turn out to be different because the investor receives or makes an additional risk-free cash flow, we can allow for constant absolute risk aversion when valuing by means of the certainty equivalent. However, we have to reject constant relative risk aversion of the investor.

3.2 Valuation Independence from Multiplicative Constants

In case of an arbitrage-free stock market the purchase of two stocks in one transaction takes place at the same price as the simultaneous purchase of two single stocks of the respective company. However, if the investor wishes to buy not one but two shares of the company, by using the certainty equivalent in case of constant absolute risk aversion, these two shares en

[9] This corresponds to the result of Kruschwitz/Löffler (2003)

[10] In case $r < 0$ inequalities (7) are valid reversely

bloc are valued less than the sum of these two single shares. For a multiplicative constant $s > 1$ with Jensen's inequality we get:[11]

$$
\begin{aligned}
\mathrm{CE}_{\exp}(s \cdot C) &= -\frac{1}{a} \cdot \ln\left(-\mathrm{E}\left(-e^{-a \cdot s \cdot C}\right)\right) \\
&= -\frac{1}{a} \cdot \ln\left(\mathrm{E}\left(e^{-a \cdot s \cdot C}\right)\right) \\
&= -\frac{1}{a} \cdot \ln\left(\mathrm{E}\left(X^{s}\right)\right) \text{ where } X \equiv e^{-a \cdot C} \\
&< -\frac{1}{a} \cdot \ln\left(\left(\mathrm{E}(X)\right)^{s}\right) \\
&= -\frac{s}{a} \cdot \ln\left(-\mathrm{E}\left(-e^{-a \cdot C}\right)\right) \\
&= s \cdot \mathrm{CE}_{\exp}(C).
\end{aligned} \tag{8}
$$

For example, this situation is relevant for an auditor who values a company in case of a management buy-out, in which it is not yet known how many managers participate in the purchase of the company's shares from the previous owners.

According to its definition, the certainty equivalent specifies the risk-free value, which provides the same utility as the considered risky cash flow. But only in special cases, the multiplication of risky cash flows leads to the same multiplication of the certainty equivalent. Therefore, in general, certainty equivalents cannot be simply treated like risk-free investments for which, in case of a perfect capital market, it is required that the doubling of the investment or the loan leads also to the doubling of cash flows. If, in valuation practice, one wishes to get results that are proportional to the company's share considered, this contradicts the assumption of constant absolute risk aversion. In case of constant relative risk aversion, however, it holds:

11 Analogously to the inequalities (7), inequality (8) is valid reversely for $0 < s < 1$

$$
\begin{aligned}
\mathrm{CE}_{\mathrm{pot}}(s\cdot C) &= \left(\gamma\cdot \mathrm{E}\left(\frac{1}{\gamma}\cdot (s\cdot C)^{1-\gamma}\right)\right)^{\frac{1}{1-\gamma}} \\
&= s\cdot\left(\gamma\cdot \mathrm{E}\left(\frac{1}{\gamma}\cdot C^{1-\gamma}\right)\right)^{\frac{1}{1-\gamma}} \\
&= s\cdot \mathrm{CE}_{\mathrm{pot}}(C) \\
\text{or}\quad \mathrm{CE}_{\log}(s\cdot C) &= \exp(\mathrm{E}(\ln(s\cdot C))) \\
&= s\cdot \exp(\mathrm{E}(\ln C)) \\
&= s\cdot \mathrm{CE}_{\log}(C).
\end{aligned} \tag{9}
$$

Thus, if in company valuation by means of certainty equivalents we require that the determined value is proportional to the share ratio, then we can now allow risk preferences with constant relative risk aversion, but have to reject risk preferences with constant absolute risk aversion.

4. Conclusion

In company valuation, value additivity is at least required with regard to additive and multiplicative constants for the sake of financial consistency in valuation practice. This requirement applies also for the certainty equivalent method and demands multiplicativity and additivity properties, accordingly. Within the class of HARA functions the two conditions imply both, constant absolute and constant relative risk aversion. These two conditions are simultaneously fulfilled only if risk aversion is inadmissibly equal to zero, i.e. in case of risk neutrality.[12]

If value additivity is required, i.e. a valuation which is independent from the investor's initial wealth and proportional to the share, company valuation is not feasible by means of certainty equivalents unless investors are risk neutral. However, then there is no need to determine any certainty equivalent.

[12] This corresponds to the result of Kürsten (2002)

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